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EINSTEIN, CONFORMALLY FLAT AND SEMI-SYMMETRIC SUBMANIFOLDS SATISFYING CHEN'S EQUALITY

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ABSTRACT

In a recent paper, B. Y. Chen proved a basic inequality between the intrinsic scalar invariants $\inf K$ and τ of M^n, and the extrinsic scalar invariant $|H|$, being the length of the mean curvature vector field H of M^n in $\mathbb{E}^m$. In the present paper we classify the submanifolds M^n of $\mathbb{E}^m$ for which

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the basic inequality actually is an equality, under the additional assumption that M^n satisfies some of the most primitive Riemannian curvature conditions, such as to be Einstein, conformally flat or semi-symmetric.

1. Introduction

Let M^n be an n-dimensional submanifold of a Euclidean space $\mathbb{E}^m$ of dimension $m = n+p, p \geq 1, n \geq 2$. Let g be the Riemannian metric induced on M^n from the standard metric on $\mathbb{E}^m$, ∇ the corresponding Levi Civita connection on M^n, and R, S and τ respectively the **Riemann–Christoffel curvature tensor**, the **Ricci tensor** and the **scalar curvature** of M^n. We use the sign convention given by $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ and the normalization of the scalar curvature given by $\tau = \sum_{i,j=1}^n K(e_i \wedge e_j)$ where K denotes the **sectional curvature** and $e_i \wedge e_j$ is the plane section of TM^n spanned by the vectors e_i and e_j for $(i \neq j)$ of an orthonormal tangent frame field $e_1, \ldots, e_n$ on M^n.

Consider the real function $\inf K$ on M^n defined for every $p \in M$ by

$$(\inf K)(p) := \inf\{K(\pi) | \pi \text{ is a plane in } T_pM^n\}.$$

Note that since the set of planes at a certain point is compact, this infimum is actually a minimum. Then B. Y. Chen proved the following basic inequality between the **intrinsic** scalar invariants $\inf K$ and τ of M^n, and the **extrinsic** scalar invariant $|H|$, being the length of the **mean curvature** vector field H of M^n in $\mathbb{E}^m$.

LEMMA ([1]): *Let $M^n, n \geq 2$, be any submanifold of $\mathbb{E}^m$, $m = n+p$, $p \geq 1$. Then*

$$(*) \qquad \inf K \geq \frac{1}{2}\{\tau - \frac{n^2(n-2)}{n-1}|H|^2\}.$$

Equality holds in $(*)$ at a point x if and only if with respect to suitable local orthonormal frames $e_1, \ldots, e_n \in T_xM^n$ and $e_{n+1}, \ldots, e_{n+p} \in T_x^\perp M^n$, the Weingarten maps A_t with respect to the normal sections $\xi_t = e_{n+t}$, $t = 1, \ldots, p$ are given by

$$A_1 = \begin{pmatrix} a & 0 & 0 & 0 & \cdots & 0 \\ 0 & b & 0 & 0 & \cdots & 0 \\ 0 & 0 & \mu & 0 & \cdots & 0 \\ 0 & 0 & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \mu \end{pmatrix}, \quad A_t = \begin{pmatrix} c_t & d_t & 0 & \cdots & 0 \\ d_t & -c_t & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad (t > 1),$$

where $\mu = a + b$. For any such frame, $\inf K(p)$ *is attained by the plane* $e_1 \wedge e_2$.

Remark 1: As stated above, this is only the special case of Chen's lemma in case that the ambient space is Euclidean. A similar inequality also holds for any real space form as ambient space [1,2], as well as for totally real submanifolds in any complex space form and C-totally real submanifolds in any Sasakian space form [3].

Remark 2: For dimension $n = 2$, $(*)$ is trivially satisfied.

Remark 3: If we choose an orthonormal basis in the following, we always take a basis as in the Lemma. Although the plane $e_1 \wedge e_2$ is not always uniquely determined, the sectional curvature $K(e_1 \wedge e_2)$ is well defined, but depends in general only continuously on the point, as follows easily from the fact that $K(e_1 \wedge e_2) = \inf K$.

The purpose of the present paper is to study the submanifolds M^n of $\mathbb{E}^m$ for which the basic inequality actually is an **equality**, under the additional assumption that M^n satisfies some of the most primitive Riemannian curvature conditions, such as to be **Einstein, conformally flat** or **semi-symmetric.** For the computations in the subsequent sections, we need the following formulas for $K_{rs} = K(e_r \wedge e_s)$:

$$K_{12} = ab - \sum_{t=2}^{n}(c_t^2 + d_t^2), \quad K_{1j} = a\mu, \quad K_{2j} = b\mu, \quad K_{ij} = \mu^2,$$

where $i, j > 2$. Furthermore, $R(e_i, e_j)e_k = 0$ if i, j and k are mutually different.

2. Einstein submanifolds satisfying Chen's equality

THEOREM 1: *Let $M^n, n \geq 3$, be a submanifold of $\mathbb{E}^m$ satisfying Chen's equality. Then M^n is Einstein if and only if M^n is a totally geodesic n-plane in $\mathbb{E}^m$.*

Proof: For a submanifold M^n satisfying the equality in $(*)$, we see that the Ricci curvatures of M^n are given by

$$\begin{aligned}
\mathrm{Ric}_1 &= (n-2)a\mu + K_{12}, \\
\mathrm{Ric}_2 &= (n-2)b\mu + K_{12}, \\
\mathrm{Ric}_3 &= \cdots = \mathrm{Ric}_n = (n-2)\mu^2,
\end{aligned}$$

and $S(e_i, e_j) = 0$ is $i \neq j$.

Now suppose that M^n is Einstein. Then from $\mathrm{Ric}_1 = \mathrm{Ric}_2$ we derive that $\mu = 0$ or $a = b$. In case $\mu = 0$, from $\mathrm{Ric}_1 = \mathrm{Ric}_3 = 0$ we get that M is totally geodesic. In case $a = b$, then $\mu = a + b = 2a$, and from $\mathrm{Ric}_1 = \mathrm{Ric}_3$ we find that $(2n-5)a^2 + \sum_{t=2}^{p}(c_t^2 + d_t^2) = 0$. Since moreover $n \geq 3$, this implies that also in this case $a = b = \mu = c_t = d_t = 0$, i.e. that M^n is totally geodesic in $\mathbb{E}^m$. ∎

3. Conformally flat submanifolds satisfying Chen's equality

We recall that a hypersurface M^n in $\mathbb{E}^{n+1}$ is called **k-quasi-umbilical**, if at each point it has a principal curvature of multiplicity $\geq n - k$ [9]. Usually 1-quasi-umbilical hypersurfaces are simply called quasi-umbilical. From the Lemma it is clear that hypersurfaces which satisfy Chen's equality are special examples of 2-quasi-umbilical hypersurfaces. From the Lemma and from Proposition 3 of [6], we obtain at once the following.

PROPOSITION 2: *Let M^n, $n \geq 3$, be a hypersurface in $\mathbb{E}^{n+1}$ satisfying Chen's equality. Then M^n is quasi-umbilical if and only if M^n is a hyperplane, a spherical hypercylinder $S^{n-1} \times \mathbb{R}$ or a round hypercone, or, in case $n = 3$, M^n is the hypersurface of revolution obtained by revolving in $\mathbb{E}^4$ the planar curve which is the graph of the function ϕ given by $\phi(x) = h^{-1}(x)$ for $x < 0$, $\phi(0) = 1$, $\phi(x) = h^{-1}(-x)$ for $x > 0$ where*

$$h(x) = \int_1^x \frac{dt}{\sqrt{t^{-4} - 1}}, x \in]0, 1[.$$

For dimension $n > 3$, by a result of E. Cartan, a hypersurface M^n in $\mathbb{E}^{n+1}$ is quasi-umbilical if and only if it is **conformally flat**, i.e. by a result of H. Weyl if the conformal curvature tensor $C = 0$. If $n = 3$, then $C = 0$ automatically, and then the above equivalence no longer holds [8].

COROLLARY 3: *Let $M^n, n > 3$, be a hypersurface in $\mathbb{E}^{n+1}$ satisfying Chen's equality. Then M^n is conformally flat if and only if M^n is a hyperplane, a spherical hypercylinder $S^{n-1} \times \mathbb{R}$ or a round hypercone.*

As $(0,4)$-tensor, Weyl's conformal curvature tensor C is defined by

$$\begin{aligned} C(X,Y;Z,W) = R(X,Y;Z,W) - \tfrac{1}{n-2}\{&S(X,W)g(Y,Z) + S(Y,Z)g(X,W) \\ &- S(X,Z)g(Y,W) - S(Y,W)g(X,Z)\} \\ &+ \tfrac{\tau}{(n-1)(n-2)}\{g(X,W)g(Y,Z) - g(X,Z)g(Z,W)\}. \end{aligned}$$

THEOREM 4: *Let $M^n, n > 3$, be a submanifold of $\mathbb{E}^m$ which satisfies Chen's equality. Then M^n is conformally flat if and only if* $\inf K = 0$.

Proof: This follows straightforward since the only components of C which are not yet trivially equal to zero are the components like C_{1221}, C_{1331}, C_{2332}, $C_{3443}, \ldots$ and they are all equal to a constant multiple of K_{12}. ∎

The **conharmonic curvature** $(0,4)$-tensor K of M^n is defined by [7]

$$K(X,Y;Z,W) = R(X,Y;Z,W) - \tfrac{1}{n-2}\{S(Y,Z)g(X,W) + S(X,W)g(Y,Z) - S(Y,W)g(X,Z) - S(X,Z)g(Y,W)\},$$

and, for $n > 3$, M^n is conharmonically related to a Euclidean space, or M^n is called **conharmonically flat** if and only if $K = 0$ [7]. It is easy to see that M^n is conharmonically flat if and only if it is conformally flat and has vanishing scalar curvature [6]. Similar to the characterization by I.M. Singer and J.A. Thorpe [10] of the 4-dimensional Einstein spaces by the fact that $K(\pi) = K(\pi^\perp)$, expressing that the sectional curvatures of M^4 for orthogonal planar sections of its tangent space are equal, the 4-dimensional conharmonically flat spaces M^4 are characterized by the fact that $K(\pi) = -K(\pi^\perp)$, see [12].

COROLLARY 5: *Let $M^n, n > 3$, be a submanifold of $\mathbb{E}^m$ which satisfies Chen's equality. Then M^n is conharmonically flat if and only if M^n is an n-plane in $\mathbb{E}^m$.*

Proof: If $K = 0$, then $C = 0$ and $\tau = 0$. Hence by Theorem 4 we have $K_{12} = 0$. Since $\tau = 2K_{12} + (n-2)(n-1)\mu^2$, it follows that $\mu = 0$, so that M is totally geodesic. ∎

4. Semi-symmetric submanifolds satisfying Chen's equality

Locally symmetric manifolds M^n are characterized by the parallelism of their curvature tensor, $\nabla R = 0$. More generally, **semi-symmetric manifolds** are characterized by the property that $R \cdot R = 0$, meaning that

$$(R(X,Y)\cdot R)(U,V)W := R(X,Y)(R(U,V)W) - R(R(X,Y)U,V)W - R(U,R(X,Y)V)W - R(U,V)(R(X,Y)W) = 0$$

for all tangent vector fields X, Y, U, V, W on M^n.

THEOREM 6: *Let M^n, $n \geq 3$, be a submanifold of $\mathbb{E}^m$ satisfying Chen's equality. Then M^n is semi-symmetric if and only if M^n is a minimal submanifold (in which case M^n is $(n-2)$-ruled), or M^n is a round hypercone in some totally geodesic subspace $\mathbb{E}^{n+1}$ of $\mathbb{E}^m$.*

Proof: By an easy computation, we find that

$$(R(e_1, e_3) \cdot R)(e_2, e_3)e_1 = a\mu(b\mu - K_{12})e_2 \tag{1}$$

and

$$(R(e_2, e_3) \cdot R)(e_1, e_3)e_2 = b\mu(a\mu - K_{12})e_1. \tag{2}$$

From the Lemma it is clear that M^n is a minimal submanifold of $\mathbb{E}^m$ if and only if $\mu = 0$, and then M is $(n-2)$-ruled and hence semi-symmetric [11]. So we can suppose that $\mu \neq 0$.

If $K_{12} = 0$, then from (1) we get that $ab = 0$. Suppose $a = 0$, then from $K_{12} = 0$ it follows that $c_t = d_t = 0$. So all shape operators vanish unless A_1. Moreover, $\mathrm{im}(h)$ is 1-dimensional and parallel, so M is essentially a hypersurface. From [4] then it follows that M is a round cone.

If $K_{12} \neq 0$, then from (1) and (2), we obtain that $a = b$. Substituting this into (1) gives us $b^2 + \sum_{t=2}^{n}(c_t^2 + d_t^2) = 0$, so $b = 0$, which contradicts $\mu \neq 0$. ∎

Following [4] and [5], a submanifold M is called **semi-parallel** if

$$R^D(X,Y)h(Z,W) - h(R(X,Y)Z,W) - h(Z,R(X,Y)W) = 0$$

for all X, Y, Z, W, where R^D is the normal curvature tensor.

Since semi-parallel submanifolds of Euclidean space are semi-symmetric, and minimal semi-parallel submanifolds of Euclidean spaces are totally geodesic (this is remarked in [4]), we have to following corollary.

COROLLARY 7: *Let $M^n, n \geq 3$, be a submanifold of $\mathbb{E}^m$ satisfying Chen's equality. Then M^n is semi-parallel if and only if M^n is totally geodesic or a round hypercone in some totally geodesic subspace $\mathbb{E}^{n+1}$ of $\mathbb{E}^m$.*

Remark 4: From [13] one may observe that in particular the hypersurfaces M^n of $\mathbb{E}^{n+1}$ which satisfy Chen's equality contain as examples several classes of hypersurfaces of pseudo-symmetry types which generalize semi-symmetry. An easy computation also shows that $R \cdot R = 0$ in Theorem 6 can be replaced by $R \cdot S = 0$.

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